Jour. Ind. Soc. Ag. Statistics
Vol. XLII, No. 2 (1990), pp. 189-200

GDTORD IN NON-ORTHOGONAL BLOCKS

RAJENDRANATH PANDA and ARCHANA DAS ROY
University of Kalyani, Kalyani

(Received : February, 1988)

SUMMARY

The concept of sequential experimentation in response surface designs is quite old. The rotatable designs conducted sequentially in two orthogonal blocks require too many design points than the non-sequential designs. For this, second order response surface designs have been conducted in non-orthogonal blocks. In this paper conditions of non-orthogonal blocking of group divisible third order rotatable designs and hence of third order rotatable designs have been derived. The method has been explained by means of an example.

Keywords : Rotatable designs, Group divisible second order rotatable design (GDSORD), Group divisible third order rotatable design (GDTORD), Non-orthogonal blocks.

Introduction

The concept of sequential experimentation was introduced by Box and Hunter [8] mainly to reduce the block size. This concept was utilised by Gardiner, Grandage and Hader [13], Draper [12], Das and Narasimham [11], Herzberg [14], Das and Dey [10] and Adhikary and Panda [4]. In Box and Hunter [8], a second order rotatable design (SORD) was performed in two orthogonal blocks such that the points of the first block formed a first order rotatable design (FORD) and the points of the two blocks together formed a SORD. Gardiner, Grandage and Hader [13] performed a third order rotatable design (TORD) in two orthogonal blocks such that the points of the first block formed a SORD.

Das and Narasimham [11] gave exhaustive list of sequential rotatable (second order and third order) designs in two orthogonal blocks. But too many central points are needed to achieve orthogonality. So Das and Dey [10] derived the conditions of a SORD performed into more than one non-orthogonal blocks. Herzberg [14] introduced the cylindrical rotatable designs and Das and Dey [9], Adhikary and Sinha [7], Adhikary and Panda [2]. [4], [5] introduced group divisible rotatable designs in order to reduce the design points by relaxing the conditions of rotatable designs. Adhikary and Panda [3] introduced the mixed order response surface designs in which the factors may not appear in the same order. It was shown that a design of lower order can be improved upon to a design of higher order by adding some points. Adhikary and Panda [4] derived the conditions of non-orthogonal blocking of group divisible response surface designs of second order. Using the designs of Adhikary and Panda [3] a SORD or group divisible second order rotatable design can be performed sequentially in non-orthogonal blocks starting from a FORD Adhikary and Panda [5] introduced the group divisible third order rotatable design (GDTORD) by relaxing the conditions of TORD.

In the present paper the conditions of non-orthogonal blocking of GDTORD have been derived in detail. It has also been shown by means of an example that how a GDTORD and hence a TORD can be performed sequentially using a mixed order designs of Adhikary and Panda [3] starting from a FORD. In the appendix a list has been given from 3 to 7 factors. Also the total number of design points and the maximum block size by this method and by the method of Das and Narasimham [11] for any design are given.

2 Model and Conditions

Let the v factors be divided into s groups as follows:

$$G_1 = (1, 2 \ldots v_1),\ G_2 = (v_1 + 1 \ldots v_1 + v_2) \ldots G_s = (v_1 + v_2 + v_{s-1} + 1 \ldots v_1 + v_2 + \ldots + v_s) \text{ where } v = v_1 + v_2 + \ldots + v_s.$$

Assuming that the experimentation has been done in m blocks, the appropriate model for GDTORD can be written as

$$y_u = \beta_0 + \sum_{\mu} \Sigma_{\mu} \beta_{i_\mu} x_{i_\mu u} + \sum_{\mu} \sum_{i_\mu < j_\mu} \sum \beta_{i_\mu j_\mu} x_{i_\mu u} x_{j_\mu u}$$

$$+ \sum_{\mu < \mu'} \sum_{\mu} \Sigma_{\mu'} \beta_{i_\mu j_{\mu'}} x_{i_\mu u} x_{j_{\mu'} u} + \sum_{\mu} \sum_{i_\mu < j_\mu < k_\mu} \Sigma \Sigma \beta_{i_\mu j_\mu k_\mu} x_{i_\mu u} x_{j_\mu u} x_{k_\mu u}$$

$$+ \sum_{\mu \neq \mu'} \sum_{i_\mu < j_\mu} \Sigma_{\mu'} \beta_{i_\mu j_\mu k_{\mu'}} x_{i_\mu u} x_{j_\mu u} x_{k_{\mu'} u} + \sum_{\mu < \mu' < \mu''} \Sigma \Sigma \; \Sigma_\mu \Sigma_{\mu'} \Sigma_{\mu''}$$

$$\beta_{i_\mu i_{\mu'} i_{\mu''}} x_{i_\mu u} x_{i_{\mu'} u} x_{i_{\mu''} u} + \sum_{w} \beta_w \left(z_{wu} - \bar{z}_w\right) + e_u' \qquad u = 1, 2 \ldots \text{(1)}$$

where β_w is the effect of wth block and $z_{wu} = 1$ if uth point belongs to the wth block and $= 0$ otherwise, $\bar{z}_w = n_w/N$ where n_w denotes the number of points in wth block. e_u's, the error components are i.i.d.r.v. with zero means and variance σ^2, $\sum_{\mu}$ denotes the sum over the factors of μth group. In addition to the conditions of GDTORD given in Adhikary and Panda [5], the points satisfy the following conditions :

(i) $$(i_\mu)b_w = \frac{1}{N} \sum_{u=1}^{n_w} x_{i_\mu u} = 0$$

(ii) $$(i_\mu j_\mu)b_w = \frac{1}{N} \sum_{u=1}^{n_w} x_{i_\mu u} x_{j_\mu u} = 0 \quad \text{for } i_\mu \neq j_\mu$$

(iii) $$(i_\mu j_{\mu'} k_{\mu''})b_w = \frac{1}{N} \sum_{u=1}^{n_w} x_{i_\mu u} x_{j_{\mu'} u} x_{k_{\mu''} u} = 0 \;\forall\; \mu, \mu', \mu''$$

(iv) $$(i_\mu^2)b_w = \frac{1}{N} \sum_{u=1}^{n_w} x_{i_\mu u}^2 = \text{constant for all factors belonging to the } \mu\text{th group.}$$

However the non-singularity condition E changes to

E : (i) $p^*_{\mu\mu} > 0$.

(ii) $p^*_{\mu\mu} \, p^*_{\mu'\mu'} - (p^*_{\mu\mu'})^2 > 0$.

where $p^*_{\mu\mu} \; [= (v_\mu + 2)\lambda_4(\mu)/v_\mu - N \sum_{w} (i^2_\mu)^2_{bw}/n_w]$,

$$\mu = 1, 2, \ldots, S; \qquad \text{(2.10)}$$

and $$p^{*}_{\mu\mu'} = \theta^{\mu\mu'} - N \sum_{w} (i^2_{\mu})_{bw}\, (i^2_{\mu'})_{bw} / n_w \qquad (2.11)$$

$$= \theta^{\mu\mu'*} \text{ (say)}$$

(iii) $$\frac{\lambda_2^{(\mu)} \lambda_6^{(\mu)}}{(\lambda_4^{(\mu)})^2} > \frac{\nu_\mu + 2}{\nu_\mu + 4}; \quad \frac{\lambda_2^{(\mu)} \theta^{\mu\mu'\mu'}}{(\theta^{\mu\mu'})^2} > \frac{\nu_{\mu}'}{\nu_{\mu}' + 2}$$

(iv) $$F_\mu - 1 = [(\nu_\mu + 4)\lambda_2(\mu)\, \lambda_6(\mu) - (\nu_\mu + 2)\, \{\lambda_4^{(\mu)}\}^2]\, x$$

$$[\, (\nu_\mu' + 2)\, \theta^{\mu\mu'\mu'}\, \lambda_2^{(\mu)} - \nu_\mu' \, (\theta^{\mu\mu'})^2] -$$

$$\nu_{\mu}' (\nu_\mu + 2)\, (\lambda_2^{(\mu)}\, \theta^{\mu'\mu\mu} - \lambda_4^{(\mu)}\, \theta^{\mu\mu'})^2 > 0$$

Let us denote by

$$N\, S\, y_{i_\mu}\, j_{\mu'}\, k_{\mu''} = \sum_{\mu} y_u\, X_{i_\mu u}\, X_{i_{\mu'} u} X_{k_{\mu''} u}.$$

Solving the normal equations Which involve $b_{i_\mu i_\mu}$'s, we have

$$2\lambda_4^{(\mu)} b_{i_\mu i_\mu} = S y_{i_\mu i_\mu} - \sum_{w} (i^2_{\mu})_{bw} - \bar{y}_w \sum_{\mu'=1}^{S} \theta^{\mu\mu'}\, \gamma_{\mu'\mu'}$$

where

$$\gamma_{u\mu} = \Sigma_\mu\, b_{i_\mu i_{\mu'}} \quad \theta^{\mu\mu*} = \lambda_4^{(\mu)} - N \sum_{w} (\, i^2_{\mu})\, b^2_w / n_w$$

$$\theta^{\mu\mu'*} = \theta^{\mu\mu'} - N \sum_{w} (i^2_{\mu})_{b_w}\, (i^2_{\mu'})_{b_w} / n_w$$

Clearly, the $\gamma_{\mu\mu'}$'s are obtained from

$$P^{*\, s \times s}\, \gamma^{s \times 1} = S^{s \times 1}$$

With $P^* = (p^*_{\mu\mu'})$, $\gamma\, (\gamma_{11}\, \gamma_{22} \ldots \gamma_{ss})$

where $p^*_{\mu\mu}$ and $p^*_{\mu\mu'}$ are given in (2.10) and (2.11). The solutions of other constants are same as in Adhikary and Panda [5].

2.1 *Construction of GDTORD on 3 factors in Non-orthogonal blocks starting from a FORD on 3 factors*

Block 1. FORD

Take 4 points obtained from $\frac{1}{2}$-replicate of 2^3 expt. With levels $\pm a$ and indentity equation $I = 123$. Also take 6 points otained from $S_i(b)$, $i = 1, 2, 3$ with $b^2 = 1.3534a^2$ where

$$S_i\,(\mathrm{b}) : 2 \times v = \begin{bmatrix} 0 \ldots & 0 \;\; b & 0 \ldots 0 \\ 0 \ldots & S - b & 0 \ldots 0 \end{bmatrix}$$

ith Column contains the elements $\pm\, b$ and elements of all other columns are zero. Thus block 1 contains 10 points. These 10 points satisfy the conditions of FORD.

Blocks 1 and 2 : GDSORD, $G_1 : (1)$, $G_2 : (2)$, $G_3 : (3)$.

The factors are divided into three groups such that each group consists only one factor.

In the second block we add points such that the factor in a group appears in second order. That is, the points of the two blocks satisfy the condition of GDSORD. In the second block 4 points obtained from $\frac{1}{2}$ replicate' of 2^3 expt. with levels $\pm\, a$ and indentify equation $I = -123$ are taken.

So total number of points of blocks 1 and 2 is 14.

Here $\mu = 3$, $i_1 = 1$, $i_2 = 3$, $N = 14$, $n_1 = 10$, $n_2 = 4$

For these points, we have

B (i) : $\Sigma\ x_{ju}^2 = 10.7068a^2$, $i \in G_\mu$, $\mu = 1, 2, 3$.

B (ii) : $\Sigma\, x_{iu}^4 = 11.6634a^4$, $i \in G_\mu$, $\mu = 1, 2, 3$.

C : $\Sigma\, x_{iu}^2\, x_{ju}^2 = 8a^4$, $i \in G_\mu$, $j \in G_{\mu'}$, $\mu \neq \mu' = 1, 2, 3$.

$3\lambda_4^{(\mu)} = 0.8331a^4$, $\alpha = 1, 2, 3$.

$(i_\mu^2)_{b1} = 0.479a^2$, $(i_\mu^2)_{b2} = 0.2857a^2$, $\mu = 1, 2, 3$.

$\Gamma^*_{\mu\mu} = 0.2262a^4 \ \forall\ \mu$, $\theta_{\mu\mu'} = 0.5714a^4 \ \forall\ \mu \neq \mu' = 1, 2, 3$

$p^{*}_{\mu\mu'} = 0.0355a^4$

$p^{*}_{\mu\mu}\, p^{*}_{\mu'\mu'} - (p_{\mu\mu'})^2 = 0.0499a^4 > 0.$

Thus all the conditions of GDSORD in 2 non-orthogonal blocks are satisfied.

Blocks 1 to 3 : SORD-GDTORD, $G_1 : (1)$, $G_2 : (2)$, $G_3 : (3)$

In block 3 we add points such that the factors of the first group appears in second order while the factors or other two groups appear in third order. That is, the contents of the block 1 to 3 satisfy the conditions of SORD-GDTORD.

For this we take 12 points given by $s_{12}(c)$, $s_{13}(c)$, $s_{23}(c)$ with $c_2 = 1.5874a^2$
where

$$S_{ij}(c) : 4 \times v = \begin{bmatrix} 0 \ldots 0 & C & 0 \ldots 0 & C & 0 \ldots 0 \\ 0 \ldots 0 & C & 0 \ldots 0 & -C & 0 \ldots 0 \\ 0 \ldots 0 & -C & 0 \ldots 0 & C & 0 \ldots 0 \\ 0 \ldots 0 & -C & 0 \ldots 0 & -C & 0 \ldots 0 \end{bmatrix}$$

That is, elements of ith and jth columns are $\pm C$ and elements of all other columns are zero.

Total number of points from 1st block to 3rd block is 26. With these 26 points all the conditions of SORD-GDTORD are satisfied.

Blocks 1 to 4 : GDTORD, $G_1 : (1)$. $G_2 : (2, 3)$.

Here we divide the factors into two groups, last group consists two factors. We add points such that the factors of the groups appear in third order. That is, points of these 4 blocks satisfy the conditions of GDTORD.

Block 4 contains 4 points given by $s_i(d)$, $i = 2, 3$. With $d^2 = 3.3478a^2$. Total number of points upto block 4 is 30.

It can be easily seen that for these 30 design points, conditions D, $D_1(i)$, $D_1(ii)$ and E, E_1 are satisfied.

Blocks 1 to 5 : TORD

In block-5 points are added such that combining all the five blocks we get a TORD.

For this we require two points given by

$$S_i(d) \quad \text{where} \quad d^2 = 3.3478a^2.$$

Thus a TORD on 3 factors with 32 design points can be performed sequentially in 5 blocks having the maximum block size 12. It is to be noted that the number of points and the maximum block size of the sequential design in two orthogonal blocks on 3 factors by Das and Narasimham are respectively 53 and 27.

ACKNOWLEDGEMENT

The authors are grateful to the referee for his valuable suggestion.

REFERENCES

[1] Adhikary, B. and Panda, R. N. (1977) : Restricted Kronecker product method of constructing second order rotatable designs, *Cal. Stat. Assoc. Bull.* **26** : 61.

[2] Adhikary, B. and Panda, R. N. (1980) : *Construction of Group Divisible Rotatable Designs.* Lecture notes on Mathematics 885, combinatories and graph theory. Proceedings, Calcutta, 1980, Springer-Verlag, Berlin, 171.

[3] Adhikary, B. and Panda, R. N. (1982) : On some mixed order response surface designs useful in sequential experiments, *Cal. Stat. Assoc. Bull.* **31** : 27.

[4] Adhikary, B. and Panda, R. N. (1983) : On group divisible response surface designs, *J. Statist. Plan. Inf.* **7** : 387.

[5] Adhikary, B. and Panda, R. N. (1984) : Group divisible third order rotatable designs (GDTORD), *Sankhya, B.* **46** : 135.

[6] Adhikary, B. and Panda, R. N. (1986) : *Construction of Nearly D-efficient Third Order Rotatable Designs Using PBIB Designs.* Symposium on Optimization, Design of Experiments and graph theory, I.I.T. Bombay, India.

[7] Adhikary, B. and Sinha, B. K. (1976) : On group divisible rotatable designs, *Cal. Stat. Assoc. Bull.* **25** : 79.

[8] Box, G. E. P. and Hunter, J. S. (1957) : Multifactor experimental designs for exploring response surfaces, *Ann, Math. Stat.* **28** : 195.

[9] Das, M. N. and Dey, A. (1967) : Group divisible rotatable designs, *Ann. Inst. Statist. Math.* **19** : 337-347.

[10] Das, M. N. and Dey, A. (1970) : On blocking second order rotatable designs, *Cal. Stat. Assoc. Bull.* **19** : 75,

[11] Das, M. N. and Narasimham, V. L. (1962) : Construction of rotatable designs through balanced incomplete block designs, *Ann. Math. Stat.* **33** : 1421-39.

[12] Draper, N. R. (1962) : Third order rotatable designs in three factors, *Technometrics* **4** : 219-234.

[13] Gardiner, G.A., Grandage, A. H. P. and Hader, R. U. (1959) : Third order rotatable designs for exploring response surface, *Ann. Math. Stat.* **30** : 1082-1906.)

[14] Herzberg, A. M. (1964) : Two third order rotatable designs in four dimensions, *Ann. Math. Stat.* **35** : 445.

APPENDIX

LIST OF SEQUENTIAL THIRD ORDER ROTATABLE DESIGNS IN NON ORTHOGONAL BLOCKS

No. of factors	*Type of the design with the grouping of the factors*	*Nature of the design points in the block*	*No. of points in the block and solution in terms of a^2*	*The total no. of design points of sequential design by this procedure and the corresponding sequential design by Das and Narasimham (1962)*	*Maximum block size of the sequential design by this procedure and the corresponding sequential design by Das and Narasimham (1962)*
1	*2*	*3*	*4*	*5*	*6*
3	Block 1 FORD, G : (123)	$\frac{1}{2}$ of 2^3 expt. levels $\pm a$, $I = 123$, $s_i (b)$ $i = 1, 2, 3$	10 $b^2 = 1.3534\ a^2$		
	Blocks 1 and 2 GDSORD, G_1 : (1) G_2 : (2) G_3 : (3)	Block 2 $\frac{1}{4}$ of 2^3 expt. levels $\pm a$, $I = -123$	4	32/(40 + 13)	12/(22 + 5)

1	(2)	(3)	(4)	(5)	(6)
	Blocks 1 to 3 SORD-GDTORD $G_1 : (1)$, $G_2 : (2)$, $G_3 : (3)$	Block 3 s_i, (c) $(ij) = (12, 13, 23)$	12 $C^2 = 1.5874a^2$		
	Blocks 1 to 4 GDTORD, $G_1 : (1)$, $G_2 = (23)$	Block 4 $s_2\ (d)$, $s_3\ (d)$	4 $d^2 = 3.3478a^2$		
	Blocks 1 to 5 TORD	Blocks 5 $s_1\ (d)$	2 $d^2 = 3.3478a^2$		
4	Block 1 FORD. $G : (1234)$	$\frac{1}{2}$ of 2^4 expt. levels $\pm\ a$ $I = 1234$	8		
	Blocks 1 and 2 FORD-GDSORD, $G_1 : (12)$ $G_2 : (3)$, $G_3 : (4)$	Block 2 $\frac{1}{2}$ of 2^4 expt. levels $\pm\ a$, $I = -1234$, $s_3\ (b_1)$, $s_4\ (b_1)$	12 $b_1^2 = 4\ a^2$		
	Blocks 1 to 3 GDSORD, G_μ, $\mu = 1, 2, 3, 4$	Block 3 $s_1\ (b)$, $s_2\ (b)$	4 $b^2 = 4a^2$		

1	2	3	4	5	6
	Blocks 1 to 4 GDTORD, G_μ, $\mu = 1, 2, 3, 4$	Block 4 $s_{12}(c)$, $s_{13}(c)$, $s_{14}(c)$, $s_{23}(c)$, $s_{24}(c)$	20 $C^2 = 1.5874a^2$		
				72/(72 + 4)	20/(48 + 4)
	Blocks 1 to 5 GDTORD, $G_1 : (12)$ $G_2 : (3)$, $G_4 : (4)$	Block 5 $s_{12}(c)$, $s_{13}(c)$, $s_{14}(c)$, $s_{23}(c)$, $s_{24}(c)$	20 $C^2 = 1.5874\ a^2$		
	Blocks 1 to 6 TORD	Block 6 $s_{34}(c)$, $s_{34}(c)$	8 $C^2 = 1.5874\ a^2$		
5	Block 1 FORD	$S_2(c)$. $s_3(c)$, $s_4(b)$, s_5 (b)	8 $b^2 = 3.2474\ a^2$ $c^2 = 1.2059\ a^2$		
	Bloks 1 and 2 GDSORD, G^{μ}, $\mu = 1,2.3,4,5$	Block 2 a — BIBD (5, 10, 6, 3, 3, 1) $\times 2^2$ where 2^2 is a fraction of 2^8 expt, with $I = 123$	40		

(1)	(2)	(3)	(4)	(5)	(6)
	Blocks 1 to 3 GDSORD-TORD G_{μ}, $\mu = 1, 2, 3$ $G_4 : (4, 5)$	Block 3 a — BIBD $(5, 10, 6, 3, 3, 1) \times 2^2$ where 2^2 is a fraction of 2^3 expt. with $I = -123$ $s_4(c)$, $s_5(c)$	44 $c^2 = 1.2059\, a^2$	100/(192 + 10)	44/120
	TORD	$s_1(b)$, $s_2(b)$, $s_3(b)$ $s_1(c)$	8 $b^2 = 3.2474\, a^2$ $c^2 = 1.2059\, a^2$		
6	Block 1 FORD	Block 1 a — PBIBD $(6, 3, 2, 4, 2, 1) \times 2^4$	48		
	Blocks 1 and 2 GDSORD, G_{μ}, $\mu = 1, 2, \ldots 6$	Block 2 a — PBIBD $(6, 8, 4, 3, 0, 2) \times 2^3$	64	244/(260 + 27)	100/211
	Blocks 1 to 3 GDTORD, G^{μ}, $\mu = 1, 2 \ldots 6$	a — PBIBD $(6, 8, 4, 3, 0, 2) \times 2^3$ a — PBIBD $(6, 3, 1, 2, 1, 0) \times 2^2$ and repeat it once $s_i(f)$, $i = 1, 2, \ldots 6$	100 $f^2 = 4\, a^2$		

1	2	3	4	5	6
	TORD	a — PBIBD (6, 3, 1, 2, 1, 0) $x2^2$ and respeat it once	24		
7	Block 1	a — BIBD (7, 3, 3, 7, 1) $\times 2^3$ where 2^2 is a fraction of 2^3 expt. with identity equation $I = 123$	28		
	Block 1 and 2 FORD — SORD, G_1 : (1234) G_2 : (567)	Block 2 a — BIBD (7, 3, 3, 7, 1) $\times 2^2$ where 2^2 is a fraction of 2^3 expt. with identity equation $I = -123$.	34 $b^2 = 4a^2$	238/(238 + 42)	64/(126 + 42)
	Blocks 1 to 3 GDSORD. G_1 : (1234) G_2 : (567)	Block 3 a — BIBD (7, 3, 3, 7, 1) $\times 2^3$ $s_i(b)$, $i = 1, 2, 3, 4$	64 $b^2 = 4a^2$		
	Blocks 1 to 4 GDTORD, G_μ, $\mu = 1, 2, \ldots, 7$	Block 4 a — BIBD (7, 4, 4, 7, 2) $\times 2^3$	56		
	Blocks 1 to 5 TORD	a — BIBD (7, 4, 4, 7, 2) $\times 2^3$	56		